

INTERSECTION NUMBER AND STABILITY OF SOME INSCRIBABLE GRAPHS

JINSONG LIU AND ZE ZHOU

ABSTRACT. A planar graph is inscribable if it is combinatorial equivalent to the skeleton of an inscribed polyhedron in the unit sphere. For an inscribable graph, if in its combinatorial equivalent class we could also find a polyhedron inscribed in each convex surface sufficiently close to the unit sphere \mathbb{S}^2 , then we call such an inscribable graph a stable one.

By combining the Teichmüller theory of packings with differential topology method, in this paper we shall investigate the stability of some inscribable graphs.

Mathematics Subject Classifications (2000): 51M20, 51M10, 52C26.

Keywords: inscribable graph, stability, intersection number, circle pattern.

0. INTRODUCTION

A graph is called planar if it can be embedded in the unit sphere \mathbb{S}^2 . And a planar graph is called inscribable if it can be realized as the skeleton of the convex hull of a set of finite points lying over the unit sphere. In the book [19], the Swiss mathematician Jakob Steiner asked for a combinatorial characterization of those inscribable graphs. To be specific, in which cases does a polyhedral graph (the skeleton of a polyhedron) can be combinatorially equivalent to the skeleton of a convex polyhedron inscribed in the sphere?

This seems to be a rather intractable problem. In fact, it's almost a hundred years later when Steinitz [20] found an example of "non-inscribable" graph in 1927. Whereafter, more and more non-inscribable graphs are discovered. For instance, the polyhedral graph of the following singly-truncated cube is exactly the simplest non-inscribable one.

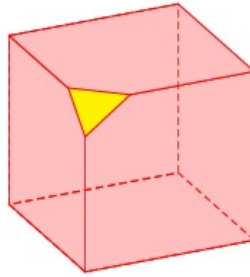


FIGURE 1. An example of non-inscribable graph

Moreover, due to the Klein Model of hyperbolic 3-space, one could regard an inscribed polyhedron as an ideal hyperbolic polyhedron. In view of such an observation, Rivin [15] then completely resolved Steiner's problem by an investigation of the geometry of ideal hyperbolic polyhedra.

For a polyhedral graph G , let G^* denote its dual graph. We call a set of edges $\Gamma = \{e_1, e_2, \dots, e_k\} \subset G$ a *prismatic circuit*, if the *dual edges* $\{e_1^*, e_2^*, \dots, e_k^*\}$ form a simple closed curve in the dual graph G^* and does not bound a face in G^* . Rivin's theorem [15] is then stated as follows.

Theorem 0.1. *A polyhedral graph $G = G(V, E)$ is of inscribable type if and only if there exists a weight w assigned to its edges set E such that:*

- (W1) *For each edge $e \in E$, $0 < w(e) < 1/2$.*
- (W2) *For each vertex v , the total weights of all edges incident to v is equal to 1.*
- (W3) *For each prismatic circuit $\gamma \subset E$, the total weights of all edges in γ is strictly greater than 1.*

Note that the condition (W2) is equivalent that the sum of the weights of edges bounding a face in the dual graph G^* is equal to 1.

In addition, for any given polyhedral graph G , Hodgson, Rivin and Smith [10] indicate that there exists a polynomial time algorithm (in the number of vertices) to decide whether it is inscribable.

These consequences are really elegant. However, a "sphere" in the real physical world often doesn't mean a standard sphere in mathematic sense. It then seems significant to go a further step to consider the stability problem of inscribable graphs. Namely, given any convex surface $\tilde{S} \subset \mathbb{R}^3$ sufficiently close to the unit sphere \mathbb{S}^2 , for an inscribable graph G , is there always a polyhedron $P_{G, \tilde{S}}$ inscribed in \tilde{S} with skeleton combinatorially equivalent to G ?

In what follows, to formulate the above question as a mathematic one, let's introduce some notions which will depict the exact meaning of "sufficiently close".

Suppose that $S_1 : \hat{\mathbb{C}} \xrightarrow{f_1} \mathbb{R}^3$, $S_2 : \hat{\mathbb{C}} \xrightarrow{f_2} \mathbb{R}^3$ are two C^k embeddings of the Riemann sphere in the 3-dimensional Euclidean space \mathbb{R}^3 . Given $\epsilon > 0$, we say S_1, S_2 are ϵ - C^k -close to each other, if the C^k -norm of every coordinate component of $f_1 - f_2$ is less than ϵ . For example, if two embedding sphere are C^3 -close to each other, it follows from the elementary surface theory that the images of S_1, S_2 and their curvatures will close to each other (see [6]). Particularly, if $\tilde{S} : \hat{\mathbb{C}} \xrightarrow{\tilde{f}} \mathbb{R}^3$ is an embedding sphere which is ϵ - C^3 -close to the unit sphere \mathbb{S}^2 for some sufficiently small $\epsilon > 0$, then the surface \tilde{S} is both strictly convex and sufficiently round.

For any given inscribable graph G , suppose that there exists an $\epsilon > 0$ such that: for any surface $\tilde{S}(\epsilon)$ which is ϵ - C^k -close to the unit sphere \mathbb{S}^2 , there is always a polyhedron $P_{G, \tilde{S}(\epsilon)}$ inscribed in $\tilde{S}(\epsilon)$ with skeleton combinatorially equivalent to G . Then we say G is C^k -stable. Recalling Rivin's result (Theorem 0.1), the problem on how to characterize an inscribable graph is equivalent to solve a system of linear inequalities. However, due to the non-openness of the solutions space of these inequalities, there may exist inscribable graph which isn't stable. That implies the stability problem of inscribable graphs wouldn't be a trivial task.

Now let $P = P(\mathcal{V}, \mathcal{E}, \mathcal{F}) \subset \mathbb{R}^3$ be given a convex polyhedron. For every vertex $v \in \mathcal{V}$, we cut a small pyramid from P by a plane which is near to v and transversal

to every edge $e \in \mathcal{E}$ emanating from v . Thus we obtain a new polyhedron P_\diamond , called the **truncated polyhedron** of P . Denote by $G(P_\diamond)$ the skeleton of P_\diamond .

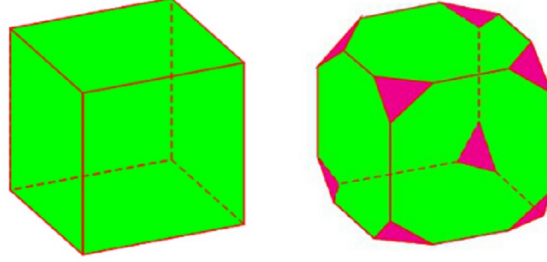


FIGURE 2. The cube and its truncated polyhedron

In this paper we shall prove

Theorem 0.2. *Let P, P_\diamond and $G(P_\diamond)$ be as above. Assume that the degree $d(v)$ of each vertex $v \in \mathcal{V}$ is odd. Then the graph $G(P_\diamond)$ is inscribable and C^1 -stable.*

In addition, for a polyhedral graph $G(P) = (\mathcal{V}, \mathcal{E}, \mathcal{F})$, let's construct a new graph $G_+(P)$ as follows. More precisely, for every edge $e \in \mathcal{E}$, we associate it with a vertex \mathbb{v}_e . Whenever two different edges $e_1, e_2 \in \mathcal{E}$ both belong to a common face $f \in \mathcal{F}$ and meet at a same vertex $v \in \mathcal{V}$, we then connect an edge from \mathbb{v}_{e_1} to \mathbb{v}_{e_2} . Thus we obtain a new graph $G_+(P)$ associated to P , which is called the **rectified graph** of the polyhedron P .

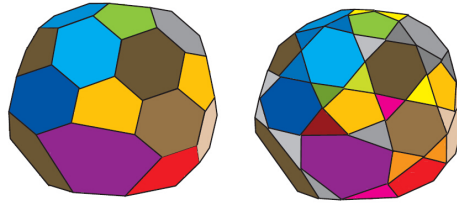


FIGURE 3. The dodecahedron and its rectified graph

For the rectified graph $G_+(P) = (\mathcal{V}_+, \mathcal{E}_+, \mathcal{F}_+)$, obviously we have

$$|\mathcal{V}_+| = |\mathcal{E}|, |\mathcal{E}_+| = 2|\mathcal{E}|, |\mathcal{F}_+| = |\mathcal{V}| + |\mathcal{F}|.$$

Furthermore, we have

Theorem 0.3. *Let $P, G_+(P)$ be as above. If $d(v)$ is odd for any vertex $v \in \mathcal{V}$, then $G_+(P)$ is inscribable and C^3 -stable.*

Given a compact strictly convex surface K , for any affine half space H^+ with $K \not\subseteq H^+$, the intersection $H^+ \cap K$ is either empty, or a point, or a topological disk. In the last case we call it a K -disk, and its boundary (in K) a K -circle. We recall that a

planar graph G is K -inscribable if there exists a polyhedron P_G inscribed in K with skeleton combinatorially equivalent to the graph G .

In terms of the above conventions, to prove Theorem 0.3 is equivalent to prove that there exists $\epsilon > 0$ such that $G_+(P)$ is $\tilde{S}(\epsilon)$ -inscribable provided that the embedding surface $\tilde{S}(\epsilon)$ is ϵ - C^3 -close to the unit sphere \mathbb{S}^2 . Recall that $G(P) = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ and $G_+(P) = (\mathcal{V}_+, \mathcal{E}_+, \mathcal{F}_+)$. To acquire such a polyhedron, we need to find the vertices set \mathcal{V}_+ such that: (1) they correspond to the tangent points of the $\tilde{S}(\epsilon)$ -circle packing realizing the graph $G^*(P)$, where $G^*(P)$ is the dual graph of the skeleton of P ; (2) if $e_1, e_2, \dots, e_k \in \mathcal{E}$ are incident to a same vertex $v \in \mathcal{V}$, then the corresponding points $\mathbb{V}_{e_1}, \mathbb{V}_{e_2}, \dots, \mathbb{V}_{e_k} \in \mathcal{V}_+$ locate in a same plane.

Hence it's necessary to prove that the intersection of these two configuration spaces is non-empty. By combining the intersection number theory from differential topology with a homotopy technique, we shall obtain the desired result. Similarly, Theorem 0.2 could be deduced by means of transversality theory.

We now briefly describe how this paper is organized. In the preliminary section we briefly give an introduction to transversality theory and intersection number theory, which will play an important role throughout this paper. In Section 3 we study the Teichmüller theory of packings, which characterizes the configuration space of K -circle packings. Section 4 is devoted to the proof of Theorem 0.3. The last section provides a geometric insight into the tangent space of another configuration. With the help of this method, we demonstrate a transversality theorem which leads to a proof of Theorem 0.2. Furthermore, we complete some details on the computation of intersection number used in Section 3.

Notational Conventions.

Through this paper, for any given set A we use the notation $|A|$ to denote the cardinality of A .

1. PRELIMINARIES

In this section, we will introduce several definitions and notations from differential topology, especially transversality and intersection number. Please refer to [7, 9] for background on these notions.

First of all, assume that M, N are two oriented smooth manifolds, and $S \subset N$ is a submanifold.

Definition 1.1. Suppose that $f : M \rightarrow N$ is a C^1 map. Given $A \subset M$, we say f is transverse to S along A , denoted by $f \pitchfork_A S$, if

$$Im(df_x) + T_{f(x)}S = T_{f(x)}N,$$

whenever $x \in A \cap f^{-1}(S)$. When $A = M$, we simply denote $f \pitchfork S$.

Let $S \subset N$ be a closed submanifold such that $dim M + dim S = dim N$. Suppose $\Lambda \subset M$ is an open subset with compact closure $\bar{\Lambda} \subset M$. Given a continuous map $f : M \rightarrow N$ such that $f(\partial\Lambda) \cap S = \emptyset$, where $\partial\Lambda = \bar{\Lambda} \setminus \Lambda$, we will define a topological invariant $I(f, \Lambda, S)$, called the intersection number between f and S in Λ .

If $f \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$ such that $f \pitchfork_\Lambda S$, then $\Lambda \cap f^{-1}(S)$ consists of finite points. For each $x \in \Lambda \cap f^{-1}(S)$, the $sgn(f, S)_x$ at x is $+1$, if the orientations on

$Im(df_{x_j})$ and $T_{f(x_j)}S$ "add up" to preserve the prescribed orientation on N , and -1 if not.

Definition 1.2. If $\Lambda \cap f^{-1}(S) = \{x_1, x_2, \dots, x_m\}$, then we define the intersection number between f and S in Λ to be

$$I(f, \Lambda, S) := \sum_{j=1}^m \text{sgn}(f, S)_{x_j}.$$

The proof of the following proposition is in the same style as that of the homotopy invariance of Brouwer degree. Please see [7, 9], or Milnor's book [14].

Proposition 1.3. Suppose that $f_i \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$, $f_i \pitchfork_\Lambda S$ and $f_i(\partial\Lambda) \cap S = \emptyset$, $i = 0, 1$. If there exists a homotopy

$$H \in C^0(I \times \bar{\Lambda}, N)$$

such that $H(0, \cdot) = f_0(\cdot)$, $H(1, \cdot) = f_1(\cdot)$, and $H(I \times \partial\Lambda) \cap S = \emptyset$, then

$$I(f_0, \Lambda, S) = I(f_1, \Lambda, S).$$

The next lemma, which helps us to manipulate the intersection number for general mappings, is a consequence of Sard's theorem [7, 9].

Lemma 1.4. For any $f \in C^0(\bar{\Lambda}, N)$ with $f(\partial\Lambda) \cap S = \emptyset$, there exists $g \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$ and $H \in C^0(I \times \bar{\Lambda}, N)$ such that

- (1) $g \pitchfork_\Lambda S$;
- (2) $H(0, \cdot) = f(\cdot)$, $H(1, \cdot) = g(\cdot)$;
- (3) $H(I \times \partial\Lambda) \cap S = \emptyset$.

The above lemma, together with Proposition 1.3, allows us to define the intersection numbers for general continuous mappings.

Definition 1.5. For any $f \in C^0(\bar{\Lambda}, N)$ with $f(\partial\Lambda) \cap S = \emptyset$, we can define the intersection number

$$I(f, \Lambda, S) = I(g, \Lambda, S).$$

where g is given as Lemma 1.4.

By Proposition 1.3, the intersection number $I(f, \Lambda, S)$ is well-defined. Furthermore, we have the following homotopy invariance property of this quantity.

Theorem 1.6. For $i = 0, 1$, suppose that $f_i \in C^0(\bar{\Lambda}, N)$ such that $f_i(\partial\Lambda) \cap S = \emptyset$. If there exists $H \in C^0(I \times \bar{\Lambda}, N)$ such that

- (1) $H(0, \cdot) = f_0(\cdot)$, $H(1, \cdot) = f_1(\cdot)$,
- (2) $H(I \times \partial\Lambda) \cap S = \emptyset$,

then we have $I(f_0, \Lambda, S) = I(f_1, \Lambda, S)$.

In particular, it immediately follows from the definition that:

Theorem 1.7. If $I(f, \Lambda, S) \neq 0$, then we have $\Lambda \cap f^{-1}(S) \neq \emptyset$.

2. TEICHMÜLLER THEORY OF PACKINGS

Given a compact strictly convex surface K , in this section we shall introduce the Teichmüller theory of K -circle packings with the same contact graph.

Roughly speaking, a K -circle (or K -disk) packing \mathcal{P} is a configuration of K -circles $\{C_v : v \in V\}$ (or disks $\{D_v : v \in V\}$) with specified patterns of tangency. The contact graph (or nerve) of \mathcal{P} is a graph $G_{\mathcal{P}}$, whose vertex set is V and an edge appears if and only if the corresponding K -circles (or K -disks) touch.

Given a planar graph $G = G(V, E)$, let's fix a vertex $v_0 \in V$ and three ordered edges $e_1, e_2, e_3 \in E$ emanating from v_0 . We call the 4-tuple $\mathcal{O} = \{v_0, e_1, e_2, e_3\}$ a combinatorial frame associated to the graph G . Suppose $\mathcal{P} = \{C_v\}$ is a K -circle packing with the contact graph $G_{\mathcal{P}} = G$. Denoting by p_1, p_2, p_3 the three tangent points corresponding to the edges e_1, e_2, e_3 , we call \mathcal{P} a normalized K -circle packing with mark $\mathfrak{M} = \{\mathcal{O}, p_1, p_2, p_3\}$.

For the compact strictly convex surface K , without loss of generality, we now assume it lies below the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ and is tangent to this plane at the point $N = (0, 0, 1)$. The point N is regarded as the "North Pole" of K . Let $h : K \rightarrow \mathbb{C} \cup \{\infty\}$ denote the "stereographic projections" with $h(N) = \infty$. Since h can be extended to a diffeomorphism between K and $\hat{\mathbb{C}}$, we then endow ∂K with a complex structure by pulling back the standard complex structure of $\hat{\mathbb{C}}$. Hence, up to conformal equivalence, we can identify K with the Riemann sphere $\hat{\mathbb{C}}$.

Given a convex polyhedron $P = P(\mathcal{V}, \mathcal{E}, \mathcal{F}) \subset \mathbb{R}^3$, we recall that $G^*(P)$ is the dual graph of the skeleton of P . Denote $G^*(P) = (V, E)$. Let us fix a disk packing $\mathcal{P}_0 = \{D_0(v)\}_{v \in V}$ on the unit sphere $\mathbb{S}^2 (\cong \hat{\mathbb{C}})$ with the contact graph $G^*(P)$. Denote $\hat{\mathbb{C}} - \bigcup_{v \in V} D_0(v) = \{I_1, I_2, \dots, I_m\}$. For each component $I \in \{I_1, I_2, \dots, I_m\}$, we call it an open interstice. Evidently, I is a topological polygon. The region I has only finitely many boundary components. And each boundary component is a piecewise smooth curve formed by finitely many circular arcs or circles. Each (maximal) circular arc or circle on the boundary ∂I belongs to a unique circle in the disk packing \mathcal{P}_0 , and therefore is marked by an element of V . The region I , together with a marking of the circular arcs or circles on its boundary by elements of V is called an *interstice* of \mathcal{P}_0 .

For each interstice I of \mathcal{P}_0 , we can define a conformal polygon as pairs $h : I \rightarrow \hat{\mathbb{C}}$, where h is a quasiconformal embedding. For details on quasiconformal mappings, please refer to Ahlfors' book [1]. The conformal polygons are considered as analogs of the conformal quadrangles.

Denote $\partial I = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$, where $\{\gamma_j\}_{1 \leq j \leq n}$ is a marking of the circular arcs or circles on its boundary. We say two such quasiconformal embeddings $h_1, h_2 : I \rightarrow \hat{\mathbb{C}}$ are Teichmüller equivalent, if the composition mapping $h_2 \circ (h_1)^{-1} : h_1(I) \rightarrow h_2(I)$ is isotopic to a conformal homeomorphism $f : h_1(I) \rightarrow h_2(I)$ such that for each side $\gamma_j \subset \partial I$, f maps $h_1(\gamma_j)$ onto $h_2(\gamma_j)$.

Definition 2.1. The Teichmüller space of I , denoted by \mathcal{T}_I , is the space of all equivalence classes of quasiconformal embeddings $h : I \rightarrow \hat{\mathbb{C}}$.

Remark 2.2. If the interstice I is k -sided, it follows from the classical Teichmüller theory that \mathcal{T}_I is diffeomorphic to the Euclidean space \mathbb{R}^{k-3} . See e.g [11].

Denote $\mathcal{T}_{G^*(P)} = \prod_{i=1}^m \mathcal{T}_{I_i}$, where $\{I_1, I_2, \dots, I_m\}$ are all interstices of the circle packing \mathcal{P}_0 . Due to Remark 2.2, we easily verify that $\mathcal{T}_{G^*(P)} \cong \mathbb{R}^{2|\mathcal{E}|-3|\mathcal{V}|}$.

Recall that a K -disk is defined as the intersection $H^+ \cap K$, where H^+ is an affine half space which intersects K . Its boundary is called a K -circle. Naturally, we call $\mathcal{P} = \{C_v : v \in V\}$ a K -circle packing, if all $C_v (v \in V)$ are K -circles. As far as these packings concerned, Liu-Zhou [12] have established the following result, which will be used in this paper as well. It's proof is a combination of the methods due to Schramm [18] and Rodin-Sullivan [16].

Lemma 2.3. *Let $K, P, G^*(P)$ and $\mathcal{T}_{G^*(P)}$ be as above. Suppose p_1, p_2, p_3 are three distinct points in K . For any*

$$[\tau] = ([\tau_1], [\tau_2], \dots, [\tau_m]) \in \mathcal{T}_{G^*(P)},$$

there exists a unique K -circle packing $\mathcal{P}_K([\tau])$ realizing the dual graph $G^(P)$ with mark $\{\mathcal{O}, p_1, p_2, p_3\}$. Moreover, it's interstice corresponding to I_i is endowed with the given complex structure $[\tau_i]$, $1 \leq i \leq m$.*

3. PROOF OF THE MAIN THEOREMS

Recall that $P = P(\mathcal{V}, \mathcal{E}, \mathcal{F}) \subset \mathbb{R}^3$ is the given convex polyhedron and P_\diamond is the corresponding truncated polyhedron. To prove the main theorems, in this section we will construct, step by step, two configurations spaces $Z_{oc}, Z(P_\diamond)$ associated to P, P_\diamond respectively.

In view of analytic geometry, we know that each affine half space $H^+ \subset \mathbb{R}^3$ can be defined as

$$H^+ = \{(x, y, u) : Ax + By + Cu + D \geq 0\} \quad (A^2 + B^2 + C^2 \neq 0).$$

Hence each H^+ is uniquely determined by the exterior unit normal vector and the intercept. In other words, it's could be depicted by a point in $\mathbb{S}^2 \times \mathbb{R}$.

Let $Z_{\mathcal{F}}$ denote the space $(\mathbb{S}^2 \times \mathbb{R})^{|\mathcal{F}|}$. Namely, a point $z_{\mathcal{F}} \in Z_{\mathcal{F}}$ gives a choice of an affine half space (or an oriented plane) for each $f \in \mathcal{F}$. $Z_{\mathcal{F}}$ will be called the \mathcal{F} -configuration space, and a point $z_{\mathcal{F}} \in Z_{\mathcal{F}}$ will be called a \mathcal{F} -configuration. For each \mathcal{F} -configuration $z_{\mathcal{F}} \in Z_{\mathcal{F}}$, we denote by $z_{\mathcal{F}}(f)$ the oriented plane corresponding to the face $f \in \mathcal{F}$.

For any $e \in \mathcal{E}$, there are $f_1, f_2 \in \mathcal{F}$ such that $f_1 \cap f_2 = e$. Let $Z_{\mathcal{F}e} \subset Z_{\mathcal{F}}$ be the set of \mathcal{F} -configurations $z_{\mathcal{F}}$ such that $z_{\mathcal{F}}(f_1)$ is parallel to $z_{\mathcal{F}}(f_2)$. Moreover, let $Z_{\mathcal{F}R} \subset Z_{\mathcal{F}}$ be the set of \mathcal{F} -configurations $z_{\mathcal{F}}$ such that the intersection

$$z_{\mathcal{F}}(f_{i_1}) \cap z_{\mathcal{F}}(f_{i_2}) \cap z_{\mathcal{F}}(f_{i_3})$$

contains more than one points for at least one triple $\{i_1, i_2, i_3\} \subset \{1, 2, \dots, |\mathcal{F}|\}$. Evidently, both $Z_{\mathcal{F}e}$ and $Z_{\mathcal{F}R}$ are closed in $Z_{\mathcal{F}}$, which implies that

$$Z_{\mathcal{F}O} = Z_{\mathcal{F}} \setminus \left(\left(\bigcup_{e \in \mathcal{E}} Z_{\mathcal{F}e} \right) \cup Z_{\mathcal{F}R} \right)$$

is open in $Z_{\mathcal{F}}$. Hence it's a manifold with the same dimension as $Z_{\mathcal{F}}$.

Let Z denote the space $Z_{\mathcal{F}O} \times \mathbb{R}^{|\mathcal{E}|} \times \mathbb{R}^{|\mathcal{E}|}$. Namely, a point $z \in Z$ gives a choice of a half space (or an oriented plane) for each $f \in \mathcal{F}$, and a choice of two points corresponding to the vertices $v_1, v_2 \in \mathcal{V}$ in the line $z_{\mathcal{F}}(f_1) \cap z_{\mathcal{F}}(f_2)$, where $f_1 \cap f_2 = v_1 v_2 = e \in \mathcal{E}$. Similarly, we call Z the configuration space. In addition, a point $z \in Z$ will be called a configuration.

For a configuration $z \in Z$, here and hereafter we simply denote by $z(f)$ the oriented plane corresponding to the face $f \in \mathcal{F}$. Moreover, if $f_1 \cap f_2 = e \in \mathcal{E}$, then we denote by $z(ve)$ the point in $z(f_1) \cap z(f_2)$ corresponding to the vertex $v \in \mathcal{V}$.

Now let $Z_{oc} \subset Z$ denote the set of configurations z such that $z(ve_i), z(ve_j), z(ve_k)$ are not collinear whenever e_i, e_j, e_k are three distinct edges incident to the same vertex $v \in \mathcal{V}$. Obviously, Z_{oc} is open in Z . Hence, Z_{oc} is a manifold with the same dimension as Z . More precisely,

$$\dim Z_{oc} = \dim Z = 3|\mathcal{F}| + 2|\mathcal{E}|.$$

For any $v \in \mathcal{V}$, suppose that $e_1, e_2, \dots, e_{d(v)}$ are all edges of P incident to the vertex v , where $d(v)$ is the degree of v . Denote by $Z_v \subset Z_{oc}$ the set of configurations z such that $z(ve_1), z(ve_2), \dots, z(ve_{d(v)})$ belong to the same plane. Define

$$Z(P_\diamond) = \cap_{v \in \mathcal{V}} Z_v.$$

In some cases, a configuration $z \in Z(P_\diamond)$ would correspond to a polyhedron in \mathbb{R}^3 combinatorially equivalent to P_\diamond . However, it's worth pointing out that there do exist configurations corresponding to other intricate geometric patterns as well. Aside from these complexity, we have:

Lemma 3.1. $Z(P_\diamond)$ is a closed submanifold of Z_{oc} with dimension $\dim Z(P_\diamond) = 3|\mathcal{E}| + 6$.

Proof. As above, let $e_1, e_2, \dots, e_{d(v)}$ be the edges of the polyhedron P emanating from v . For each $i = 1, 2, \dots, d(v)$, denote $z(ve_i) = (x_i, y_i, u_i)$.

Consider the matrix

$$\begin{pmatrix} x_1 & y_1 & u_1 & 1 \\ x_2 & y_2 & u_2 & 1 \\ x_3 & y_3 & u_3 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{d(v)} & y_{d(v)} & u_{d(v)} & 1 \end{pmatrix}$$

Then $z(ve_1), z(ve_2), \dots, z(ve_{d(v)})$ belong to the same plane if and only if the rank of the above matrix is less than 4. Equivalently, the determinant

$$R(z(ve_{i_1}), z(ve_{i_2}), z(ve_{i_3}), z(ve_{i_4})) = \begin{vmatrix} x_{i_1} & y_{i_1} & u_{i_1} & 1 \\ x_{i_2} & y_{i_2} & u_{i_2} & 1 \\ x_{i_3} & y_{i_3} & u_{i_3} & 1 \\ x_{i_4} & y_{i_4} & u_{i_4} & 1 \end{vmatrix} = 0.$$

for each subset $\{i_1, i_2, i_3, i_4\} \subset \{1, 2, \dots, d(v)\}$.

In view of the definition of the space Z_{oc} , it follows that $z(ve_{j_1}), z(ve_{j_2}), z(ve_{j_3})$ aren't collinear for any three different subscripts $\{j_1, j_2, j_3\} \subset \{1, 2, \dots, d(v)\}$. That implies that 0 is the regular value of the smooth function $R(ve_{i_1}, ve_{i_2}, ve_{i_3}, ve_{i_4})$. Owing to the regular value theorem [9], $Z(P_\diamond)$ is then a closed submanifold of Z_{oc} . Moreover, we have

$$\dim Z(P_\diamond) = 3|\mathcal{F}| + 2|\mathcal{E}| - \left(\sum_{v \in \mathcal{V}} d(v) - 3 \right) = 3|\mathcal{F}| + 2|\mathcal{E}| - (2|\mathcal{E}| - 3|\mathcal{V}|) = 3|\mathcal{E}| + 6,$$

where the last identity comes from Euler's formula. \square

Let K be a given compact strictly convex surface. Choose a combinatorial frame \mathcal{O} for $G^*(P)$ and three different points p_1, p_2, p_3 in K . For each $[\tau] \in \mathcal{T}_{G^*(P)}$, from Lemma 2.3, it follows that there is a unique normalized K -circle packing $\mathcal{P}_K([\tau])$ realizing the graph $G^*(P)$ with the mark $\mathfrak{M} = \{\mathcal{O}, p_1, p_2, p_3\}$.

Note that $G^*(P) = (V, E, F)$ is the dual graph of the polyhedral graph $G(P) = (\mathcal{V}, \mathcal{E}, \mathcal{F})$. For any $f \in \mathcal{F}$, then $f^* \in V$. For the K -circle packing $\mathcal{P}_K([\tau])$, denote by $H^+(f^*)$ the oriented plane corresponding to the vertex $f^* \in V$.

If $e \in \mathcal{E}$, then $e^* \in E$. In addition, let $p_{e^*} \in K$ be the tangent points associating with the edge $e^* \in E$. We now associate $\mathcal{P}_K([\tau])$ with a configuration $z(\tau) \in Z_{oc}$ such that $z(\tau)(f) = H^+(f^*)$ and $z(\tau)(v_1e) = z(\tau)(v_2e) = p_{e^*}$. Consequently, it gives rise to the following mapping:

$$f_{K, \mathfrak{M}} : \mathcal{T}_{G^*(P)} \longrightarrow Z_{oc} \hookrightarrow Z.$$

Furthermore, an elementary calculation gives

$$\dim Z(P_\diamond) = 3|\mathcal{F}| + 2|\mathcal{E}| - (2|\mathcal{E}| - 3|\mathcal{V}|) = 3|\mathcal{E}| + 6,$$

$$\dim \mathcal{T}_{G^*(P)} = 2|\mathcal{E}| - 3|\mathcal{V}|,$$

$$\dim \mathcal{T}_{G^*(P)} + \dim Z(P_\diamond) = 3|\mathcal{F}| + 2|\mathcal{E}| = \dim Z_{oc}.$$

These identities remind us of the intersection number theory. In order to apply this tool, it's necessary to find a proper compact set $\Lambda \subset \mathcal{T}_{G^*(P)}$ such that $f_{K, \mathfrak{M}}(\partial\Lambda) \cap Z(P_\diamond) = \emptyset$.

Given $\epsilon > 0$, we denote by $\mathcal{B}(\mathbb{S}^2, \epsilon)$ the set of compact convex surfaces which are ϵ - C^3 -close to the unit sphere \mathbb{S}^2 .

Lemma 3.2. *Assume that $d(v)$ is odd for every $v \in \mathcal{V}$. Then there exists $\epsilon > 0$ such that: Any $K \in \mathcal{B}(\mathbb{S}^2, \epsilon)$ is convex and there is a compact set $\Lambda \subset \mathcal{T}_{G^*(P)}$ such that $f_{K, \mathfrak{M}}(\partial\Lambda) \cap Z(P_\diamond) = \emptyset$.*

Proof. Due to the continuity, the above lemma will be deduced if we could prove the existence of Λ such that $f_{K, \mathfrak{M}}(\partial\Lambda) \cap Z(P_\diamond) = \emptyset$ for $K = \mathbb{S}^2$.

To simplify notations, let $f_0 = f_{\mathbb{S}^2, \mathfrak{M}}$. Note that a configuration $z \in f_0(T_{G^*(P)}) \cap Z(P_\diamond)$ corresponds to an ideal polyhedron with skeleton combinatorially equivalent to $G_+(P)$. We could consider this ideal polyhedron as a circle packing \mathcal{P}_0 on the Riemann sphere realizing $G^*(P)$. For any $v \in \mathcal{V}$, we assume that $e_1, e_2, \dots, e_{d(v)}$ are the edges of the ideal polyhedron emanating from v . Let $p_{e_1^*}, p_{e_2^*}, \dots, p_{e_{d(v)}^*}$ be the corresponding tangent points of the packing \mathcal{P}_0 . It follows that $p_{e_1^*}, p_{e_2^*}, \dots, p_{e_{d(v)}^*}$ are contained in a common plane, which implies that they are contained in a common circle C_v .

For $i \in \{1, 2, \dots, d(v)\}$, let θ_i be the dihedral angle between C_v and $C_{f_i^*}$, where $C_{f_i^*} \in \mathcal{P}_0$ is the circle that contains the tangent points $p_{e_i^*}, p_{e_{i+1}^*}$. Since $d(v)$ is odd, a simple computation then shows that

$$(1) \quad \theta_1 = \theta_2 = \dots = \theta_{d(v)} = \frac{\pi}{2}.$$

This implies that the lines $l_{e_1^*}, l_{e_2^*}, \dots, l_{e_{d(v)}^*}$ intersect at the center of the circle C_v , where $l_{e_i^*}, 1 \leq i \leq d(v)$, is the common tangent line of the circles $\{C_{f_i^*}, C_{f_{i+1}^*}\}$ in the packing \mathcal{P}_0 .

Now we assume, by contradiction, that there is not such a compact set Λ . Then there is a sequence of $[\tau]_n \in f_0^{-1}(Z(P_\diamond))$ such that the corresponding normalized packings $\mathcal{P}_n = \mathcal{P}(\tau_n)$ satisfy one of the follow two possibilities:

- As $n \rightarrow \infty$, there exists $f^* \in V$, such that the corresponding circles $\{C_{n, f^*}\}$ in the packings \mathcal{P}_n tends to a point;
- For some v^* , as $n \rightarrow \infty$, the distance of two non-adjacent arcs of the interstice $I_{v^*, n}$ of the packings \mathcal{P}_n tends to zero.

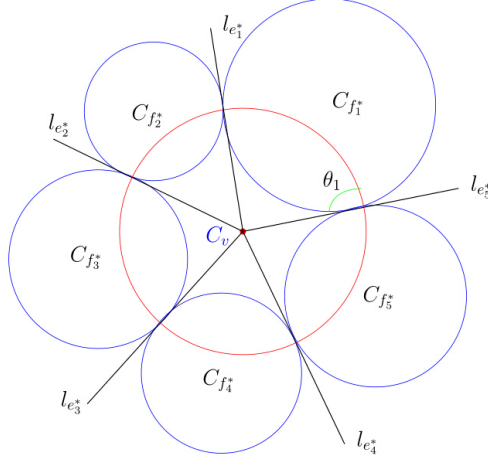


FIGURE 4.

In the first case, suppose that there exists at least one circle tending to a point. Note that any three circles with disjoint interiors can not meet at a common point. Therefore, all circles in the packing sequence \mathcal{P}_n will degenerate to points, except for at most two circles. It contradicts to our normalization conditions. We thus rule out the first possibility.

Now we turn to the second case. Noting that $[\tau]_n \in f_0^{-1}(Z(P_\diamond))$, they correspond to a sequence of ideal polyhedra P_n . Hence the tangent lines of the packings \mathcal{P}_n will separate the non-adjacent arcs. On the other hand, we have known that the sizes of all circles in \mathcal{P}_n have positive infimum. These facts tell us that the distance of such non-adjacent arcs can't tend to zero, which rules out the second possibility. \square

Remark 3.3. It's worth pointing out that Equation (1) wouldn't hold any more if $d(v)$ is even for some $v \in \mathcal{V}$. In fact, this seems to be the main obstruction on why we couldn't extend Theorem 0.2 and Theorem 0.3 to more general cases.

Assume that $K \in \mathcal{B}(\mathbb{S}^2, \epsilon)$. If we could prove $I(f_{K, \mathfrak{M}}, \Lambda, Z(P_\diamond)) \neq 0$, then Theorem 1.7 implies that $f_{K, \mathfrak{M}}^{-1}(Z(P_\diamond)) \cap \Lambda \neq \emptyset$, which proves Theorem 0.3. Recalling Theorem 1.6, to determine the intersection numbers, let's use a homotopy method.

Note that K is a given compact strict convex surface. Without loss of generality, we assume that its diameter is larger than 1. Furthermore, assume that the unit sphere \mathbb{S}^2 is internally tangent to K at the point $N = (0, 0, 1)$. Then $N = (0, 0, 1)$ could be considered as the common "North Pole" of \mathbb{S}^2 and K .

Let h_0, h_1 be the "stereographic projections" for \mathbb{S}^2, K respectively. Define a one parameter family of closed surfaces by

$$\{s \cdot h_1^{-1}(z) + (1-s) \cdot h_0^{-1}(z) : z \in \hat{\mathbb{C}}\}.$$

For each $s \in [0, 1]$, the above set is a compact strictly convex surface in \mathbb{R}^3 . Denote it by K_s . Then $\{K_s\}_{1 \leq s \leq 1}$ is a family of compact strictly convex surface joining \mathbb{S}^2 and K . Similarly, we can endow the smooth convex surface K_s with the complex structure $\hat{\mathbb{C}}$ for each $s \in [0, 1]$ by the "stereographic projection". Moreover, with

the help of Lemma 2.3, we could construct a mapping

$$f_s = f_{K, \mathfrak{M}} : \mathcal{T}_{G^*(P)} \rightarrow Z_{oc},$$

which is a homotopy from f_0 to $f_{K, \mathfrak{M}}$. Furthermore, if $K \in \mathcal{B}(\mathbb{S}^2, \epsilon)$, from Lemma 3.2 it follows that there exists $\Lambda \subset \mathcal{T}_{G^*(P)}$ such that $f_s(\partial\Lambda) \cap Z(P_\diamond) = \emptyset$ for all $s \in [0, 1]$. Furthermore, we conclude that:

Theorem 3.4. *Suppose that $d(v)$ is odd for each $v \in \mathcal{V}$. Given any $K \in \mathcal{B}(\mathbb{S}^2, \epsilon)$, then $I(f_{K, \mathfrak{M}}, \Lambda, Z(P_\diamond)) = 1$.*

Proof. Due to Theorem 1.6, it's necessary to calculate $I(f_0, \Lambda, Z(P_\diamond))$. From the following Proposition 3.5, we have $I(f_0, \Lambda, Z(P_\diamond)) = 1$. It thus completes the proof. \square

Proposition 3.5. *Suppose that $d(v)$ is odd for any $v \in \mathcal{V}$. Then $I(f_0, \Lambda, Z(P_\diamond)) = 1$*

The proof of this result is postponed to the next section.

Up to now, we have developed all the necessary results for our purpose. It's ready to prove one of the main results of this paper.

Proof of Theorem 0.3. As pointed out, it is an immediate consequence of Theorem 1.7 and Theorem 3.4, which completes the proof. \square

4. TRANSVERSALITY AND COMPUTATION OF INTERSECTION NUMBERS

It remains to prove Theorem 0.2 and Proposition 3.5. To reach this goal, we shall make use of transversality theory and a hopotopy method.

Let's employ a consequence concerning the Teichmüller theory of circle patterns. Recall that $G^*(P)$ is the dual graph of the skeleton of P . In [8], He-Liu have proved the following theorem:

Lemma 4.1. *Suppose that a weight function $w : E \rightarrow [0, \pi/2]$ satisfies the following two conditions:*

- (i) *If three distinct edges e_i^*, e_j^*, e_k^* form a simple closed loop in $G^*(P)$, then $w(e_i^*) + w(e_j^*) + w(e_k^*) < \pi$.*
- (ii) *If four distinct edges $e_i^*, e_j^*, e_k^*, e_l^*$ form a simple closed loop in $G^*(P)$, then $w(e_i^*) + w(e_j^*) + w(e_k^*) + w(e_l^*) < 2\pi$.*

For any

$$[\tau] = ([\tau_1], [\tau_1], \dots, [\tau_n]) \in \mathcal{T}_{G^*(P)},$$

there exists a unique normalized circle pattern $\mathcal{P}_w([\tau])$ with contact graph $G^*(P)$ and dihedral angle $w(e^*) : e^* \in E$. Moreover, the corresponding interstices of $\mathcal{P}_w([\tau])$ are endowed with the given complex structure $[\tau_i], 1 \leq i \leq n$.

Let W be the set of weight functions that satisfy the above conditions (i) and (ii). Lemma 4.1 implies that we can define, for each $w \in W$, a mapping $f_{w, \mathfrak{M}} : \mathcal{T}_{G^*(P)} \rightarrow Z_{oc}$ via associating every $[\tau] \in \mathcal{T}_{G^*(P)}$ with the unique normalized circle pattern realizing the complex structure $[\tau]$. More precisely, we define $f_{w, \mathfrak{M}}([\tau]) = z$, where z is the unique configuration such that: (1) $z(f)$ (we view it as an oriented plane) contains the circle C_{f^*} ; (2) $z(v_1e)$, $z(v_2e)$ are the two intersection points of $C_{f_1^*}, C_{f_2^*}$ corresponding the vertices $v_1, v_2 \in \mathcal{V}$, where $f_1 \cap f_2 = e$.

Denoting $w_0 = (0, 0, \dots, 0)$ and $w_s = sw + (1-s)w_0$, $s \in [0, 1]$, then $f_{w_s, \mathfrak{M}}$ is a homotopy from $f_0 = f_{w_0, \mathfrak{M}}$ to $f_{w, \mathfrak{M}}$. Furthermore, suppose that we have chosen

$w \in W$ sufficiently close to w_0 . By using a similar argument as in Lemma 3.2, we deduce that there exists a compact subset $\Lambda \subset \mathcal{T}_{G^*(P)}$ such that $f_{w_s, \mathfrak{M}}(\partial\Lambda) \cap Z(P_\diamond) = \emptyset$ for all $s \in [0, 1]$.

In order to calculate $I(f_{w, \mathfrak{M}}, \Lambda, Z(P_\diamond))$, it seems necessary to investigate the transversality between $f_{w, \mathfrak{M}}$ and $Z(P_\diamond)$. We thus need the following Andreev's theorem [2, 3, 6], which provide us a geometric insight into the tangent space of $Z(P_\diamond)$. Denote by \mathcal{E}_\diamond the edges set of P_\diamond . Then we have

Lemma 4.2. *Let P_\diamond be a trivalent polyhedron in \mathbb{R}^3 with a weight function $w_\diamond : \mathcal{E}_\diamond \rightarrow (0, \pi/2]$ attached to its edge set. There is a compact hyperbolic polyhedra Q_\diamond combinatorially equivalent to P_\diamond with the dihedral angle $\theta(e_\diamond)$ equal to $w(e_\diamond)$ if and only if the following conditions hold:*

- (1_a) *If three distinct edges $e_{\diamond i}, e_{\diamond j}, e_{\diamond k}$ meet at a vertex, then $w(e_{\diamond i}) + w(e_{\diamond j}) + w(e_{\diamond k}) > \pi$.*
- (2) *If $\{e_{\diamond i}, e_{\diamond j}, e_{\diamond k}\}$ is a prismatic 3-circuit, then $w(e_{\diamond i}) + w(e_{\diamond j}) + w(e_{\diamond k}) < \pi$.*
- (3) *If $\{e_{\diamond i}, e_{\diamond j}, e_{\diamond k}, e_{\diamond l}\}$ is a prismatic 4-circuit, then $w(e_{\diamond i}) + w(e_{\diamond j}) + w(e_{\diamond k}) + w(e_{\diamond l}) < 2\pi$.*

Furthermore, this polyhedron is unique up to isometries of \mathbb{B}^3 .

Recall that the Klein model of the hyperbolic 3-space \mathbb{H}^3 . In this model, \mathbb{H}^3 is identified with the interior of the unit ball $\mathbb{B}^3 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$. In addition, a geodesic in \mathbb{H}^3 corresponds to the intersection of a straight line of \mathbb{RP}^3 with \mathbb{B}^3 , and a totally geodesic plane in \mathbb{H}^3 corresponds to the intersection of a plane of \mathbb{RP}^3 with \mathbb{B}^3 . Then a convex body in \mathbb{H}^3 is represented by a convex body in \mathbb{B}^3 .

Furthermore, in terms of Bao-Banahon [4], a hyperideal polyhedron Q_{hi} is defined to be a compact convex polyhedron in \mathbb{RP}^3 whose vertices locate outside of the closed unit ball \mathbb{B}^3 and whose edges all meet \mathbb{B}^3 .

Observe that the truncated polyhedron P_\diamond is a trivalent polyhedra if and only if $G^*(P_\diamond)$ is a triangular graph, where $G^*(P_\diamond)$ is the dual graph of the skeleton of the polyhedron P_\diamond . Recall the definition of prismatic circuits given in Section 0.

By either Circle Pattern Theorem [13, 21] or Hyperideal Polyhedra Theorem [4], we have:

Lemma 4.3. *Let P_\diamond be a trivalent polyhedron in \mathbb{R}^3 with a weight function $w_\diamond : \mathcal{E}_\diamond \rightarrow [0, \pi/2]$ attached to its edges set. There is a compact hyperideal polyhedra Q_{hi} combinatorially equivalent to P_\diamond with the dihedral angle of e_\diamond equal to $w(e_\diamond)$ if and only if the following conditions hold:*

- (1_b) *If three distinct edges $e_{\diamond i}, e_{\diamond j}, e_{\diamond k}$ meet at a vertex, then $w(e_{\diamond i}) + w(e_{\diamond j}) + w(e_{\diamond k}) \leq \pi$.*
- (2) *If $\{e_{\diamond i}, e_{\diamond j}, e_{\diamond k}\}$ is a prismatic 3-circuit, then $w(e_{\diamond i}) + w(e_{\diamond j}) + w(e_{\diamond k}) < \pi$.*
- (3) *If $\{e_{\diamond i}, e_{\diamond j}, e_{\diamond k}, e_{\diamond l}\}$ is a prismatic 4-circuit, then $w(e_{\diamond i}) + w(e_{\diamond j}) + w(e_{\diamond k}) + w(e_{\diamond l}) < 2\pi$.*

This polyhedron is unique up to an element of $PO(3, 1)$, where the group $PO(3, 1)$ consists of those projective transformations of \mathbb{RP}^3 which respect the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$.

Furthermore, a vertex is located on the unit sphere if and only if the equality holds in Condition 1_b for this vertex.

Recall that the skeleton of the truncated polyhedron $G(P_\diamond) = (\mathcal{V}_\diamond, \mathcal{E}_\diamond, \mathcal{F}_\diamond)$. We call $e_\diamond \in \mathcal{E}_\diamond$ an ordinary edge if e_\diamond actually corresponds to an edge $e \in \mathcal{E}$ in the polyhedron P . Other edges of $\mathcal{E}_\diamond \setminus \mathcal{E}$ are called special edges. Without leading to

ambiguity, here and hereafter we shall not distinguish an ordinary edge $e \in \mathcal{E}$ with its corresponding edge in \mathcal{E}_\diamond .

Similarly, we can define the ordinary faces and the special faces of \mathcal{F}_\diamond . Obviously, each special face of \mathcal{F}_\diamond corresponds to a vertex of \mathcal{V} .

By using the above two lemmas, we have the following result.

Proposition 4.4. *Suppose that $d(v)$ is odd for any $v \in \mathcal{V}$. Then $f_{w,\mathfrak{M}} \pitchfork Z(P_\diamond)$.*

Proof. If $z = f_{w,\mathfrak{M}}([\tau]) \in f_{w,\mathfrak{M}}(\mathcal{T}_{G^*(P)} \cap Z(P_\diamond))$, then the configuration z is combinatorially equivalent to the truncated polyhedron P_\diamond .

From Theorem 0.1, Lemma 4.2 and Lemma 4.3, it follows that there exists an injection

$$\Psi : PO(3, 1) \times U \rightarrow Z(P_\diamond),$$

where U is the relatively open convex set of $(0, \pi/2]^{3|\mathcal{E}|}$ defined by the constraint conditions (2) and (3). Moreover, an elementary computation shows that the map Ψ is differentiable. Note that

$$\dim PO(3, 1) + \dim U = 6 + 3|\mathcal{E}| = \dim Z(P_\diamond).$$

The injectivity then tells us that there exist $(m_1, w_{\diamond 1}) \in PO(3, 1) \times U$ such that $z = \Psi(m_1, w_{\diamond 1})$ and the pushing map

$$\Psi_* : T_{m_1} PO(3, 1) \times T_{w_{\diamond 1}} U \rightarrow T_z Z(P_\diamond)$$

is a linear isomorphism.

For any ordinary edge e , denote by $v_1 e, v_2 e$ the two end points of the edge e in the truncated polyhedron (corresponding to the vertices $v_1, v_2 \in \mathcal{V}$). Moreover, for $i = 1, 2$, we define the defect curvature $k(v_i e)$ at the vertex $v_i e$ to be

$$k(v_{i,e}) = \pi - (w(e) + w(e_{v_i,1}) + w(e_{v_i,2})),$$

where $e, e_{v_i,1}, e_{v_i,2}$ are the three distinct edges incident to the vertex $v_i e$ in the truncated polyhedron P_\diamond . Note that the tangent space $T_{w_{\diamond 1}} U$ is expanded by the vectors

$$\left\{ \frac{\partial}{\partial w(e_{\diamond 1})}, \frac{\partial}{\partial w(e_{\diamond 2})}, \dots, \frac{\partial}{\partial w(e_{\diamond 3|\mathcal{E}|})} \right\}.$$

When $d(v)$ is odd for each $v \in \mathcal{V}$, it's not hard to deduce that this tangent space is equivalent to the \mathbb{R} -linear space expanded by

$$\left\{ \frac{\partial}{\partial w(e_1)}, \frac{\partial}{\partial k(v_1 e_1)}, \frac{\partial}{\partial k(v_2 e_1)}, \dots, \frac{\partial}{\partial w(e_{|\mathcal{E}|})}, \frac{\partial}{\partial k(v_1 e_{|\mathcal{E}|})}, \frac{\partial}{\partial k(v_2 e_{|\mathcal{E}|})} \right\},$$

where $\{e_1, e_2, \dots, e_{|\mathcal{E}|}\}$ are all ordinary edges of the polyhedron P_\diamond .

Since $\Psi_* : T_{m_1} PO(3, 1) \times T_{w_{\diamond 1}} U \rightarrow T_z Z(P_\diamond)$ is a linear isomorphism, we can identify $PO(3, 1)$ with the space of all marks $\mathfrak{M} = \{\mathcal{O}, p_1, p_2, p_3\}$.

Any special face of \mathcal{F}_\diamond corresponding to the vertex $v \in \mathcal{V}$ is a $d(v)$ -sided polygon. We fix $d(v) - 3$ diagonal lines emanating from the same vertex of this special face. Using these

$$\sum_{v \in \mathcal{V}} (d(v) - 3) = 2|\mathcal{E}| - 3|\mathcal{V}|$$

diagonal lines serving as bending lines, from Theorem 0.1 it follows that there is a family of ideal hyperbolic convex polyhedra $z(\theta_1, \theta_2, \dots, \theta_{2|\mathcal{E}| - 3|\mathcal{V}|})$, where $0 \leq \theta_j \leq$

$\epsilon_0, 1 \leq j \leq 2|\mathcal{E}| - 3|\mathcal{V}|$, are exterior dihedral angles. Please refer to Part I & II of the book [5]. It implies that the tangent map

$$df_{w,\mathfrak{M}} : T_{[\tau]}\mathcal{T}_{G^*(P)} \rightarrow T_z Z_{oc}$$

is differentiable and injective.

Now we assume that $\mathbf{t} \in df_{w,\mathfrak{M}}(T_{[\tau]}\mathcal{T}_{G^*(P)}) \cap T_z Z(P_\diamond)$ is a tangent vector. Then $\mathbf{t} \in T_z Z(P_\diamond)$ corresponds to an infinitesimal change of the dihedral angle $w(e_j)$ of some ordinary edge e_j , or the defect curvature $k(v_i e_j)$ of some vertex $v_i e_j \in \mathcal{V}_\diamond$, or the mark $\mathfrak{M} = \{\mathcal{O}, p_1, p_2, p_3\}$. On the other hand, $\mathbf{t} \in df_{w,\mathfrak{M}}(T_{[\tau]}\mathcal{T}_{G^*(P)})$, the mark and the dihedral angles of the ordinary edges never change. Furthermore, due to the definition, the vertices of $z(\theta_1, \theta_2, \dots, \theta_{2|\mathcal{E}| - 3|\mathcal{V}|})$ keep locating on \mathbb{S}^2 . Note that a non-trivial change on defect curvature $k(v_i e_j)$ means a deviation from $\partial \mathbb{B}^3 = \mathbb{S}^2$. Hence $\mathbf{t} = 0$, which thus completes the proof. \square

Corollary 4.5. $I(f_0, \Lambda, Z(P_\diamond)) = I(f_{w,\mathfrak{M}}, \Lambda, Z(P_\diamond)) = 1$

Proof. Owing to the rigidity of ideal hyperbolic polyhedra [15], there exists only one point in the intersection $f_{w,\mathfrak{M}}(\mathcal{T}_{G^*(P)}) \cap Z(P_\diamond)$. By Theorem 4.4, we thus show that $I(f_{w,\mathfrak{M}}, \Lambda, Z(P_\diamond)) = 1$. In view of Theorem 1.6, the corollary is complete. \square

In the end, let's prove Theorem 0.2, which is another main result of this paper.

Proof of Theorem 0.2. Without loss of generality, we assume that the unit sphere \mathbb{S}^2 is contained in the interior of the convex surface K . We shall construct a new mapping $f_{w,K,\mathfrak{M}} : \mathcal{T}_{G^*(P)} \rightarrow Z_{oc}$ associated to the data w and K .

As mentioned above, for any $[\tau] \in T_{G^*(P)}$, from Lemma 4.1, it follows that there exists a unique normalized circle pattern $\mathcal{P}(w, [\tau])$ on \mathbb{S}^2 realizes the data w and $[\tau]$ with the contact graph $G^*(P)$. Denote $\mathcal{P}(w, [\tau]) = \{C_{f^*} : f \in \mathcal{F}\}$ and let H_f^+ be the oriented plane where the circle C_{f^*} locates. Now let's define $f_{w,K,\mathfrak{M}}([\tau]) = z$ such that $z(f) = H_f^+$ and $z(v_{1,e}), z(v_{2,e})$ are exactly the two intersection points in $H_{f_1}^+ \cap H_{f_2}^+ \cap K$ when $f_1 \cap f_2 = e$. We thus construct the mapping $f_{w,K,\mathfrak{M}} : \mathcal{T}_{G^*(P)} \rightarrow Z_{oc}$. Note that $f_{w,\mathbb{S}^2,\mathfrak{M}} = f_{w,\mathfrak{M}}$ and the condition $f_{w,\mathfrak{M}} \pitchfork Z(P_\diamond)$ is stable under a slight C^1 -perturbation, we prove the statement of this theorem. \square

REFERENCES

- [1] Ahlfors, Lars V. Lectures on quasiconformal mappings. No. 10. AMS Bookstore, 1966.
- [2] Andreev, E. M. On convex polyhedra in Lobachevskii spaces. *Matematicheskii Sbornik* 123.3 (1970): 445-478.
- [3] Andreev, E. M. On convex polyhedra of finite volume in Lobachevskii space. *Sbornik: Mathematics* 12.2 (1970): 255-259.
- [4] Bao Xiliang, and Bonahon Francis. Hyperideal polyhedra in hyperbolic 3-space. *Bull. Soc. Math. France*, 2002, 130(3): 457-491.
- [5] Richard D. Canary, David Epstein and Albert Marden. *Fundamentals of hyperbolic geometry: selected expositions*. London Mathematical Society Lecture Note Series, 328. Cambridge University Press, Cambridge, 2006. xii+335 pp.
- [6] Do Carmo, Manfredo Perdigao, and Manfredo Perdigao Do Carmo. *Differential geometry of curves and surfaces*. Vol. 2. Englewood Cliffs: Prentice-Hall, 1976.
- [7] Guillemin, Victor, and Alan Pollack. *Differential topology*. Vol. 370. American Mathematical Soc., 2010.
- [8] He, Zhengxu, and Liu Jinsong. On the Teichmüller theory of circle patterns. *Trans. AMS* 365 (2013): 6517-6541.
- [9] Hirsch, Morris W. *Differential topology*. Graduate Texts in Mathematics (1976).

- [10] Hodgson, Craig D., Igor Rivin, and Warren D. Smith. A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere. *Bulletin of the American Mathematical Society* 27.2 (1992): 246-251.
- [11] Lehto Olli, Virtanen K I. *Quasiconformal mappings in the plane*. New York: Springer, 1973.
- [12] Liu, Jingsong and Zhou Ze. How many cages midscribe an egg. *arXiv:1412.5430*, 2014.
- [13] Marden, Al, and Rodin Burt. On Thurston's formulation and proof of Andreev's theorem. *Computational Methods and Function Theory* (1990): 103-115.
- [14] Milnor, John W. *Topology from the differentiable viewpoint*. Princeton University Press, 1997.
- [15] Rivin, Igor. A characterization of ideal polyhedra in hyperbolic 3-space. *Annals of mathematics* (1996): 51-70.
- [16] Rodin, Burt, and Sullivan Dennis. The convergence of circle packings to the Riemann mapping. *Journal of Differential Geometry* 26.2 (1987): 349-360.
- [17] Roeder, Roland KW, John H. Hubbard, and William D. Dunbar. Andreev's theorem on hyperbolic polyhedra. In *Annales de l'institut Fourier*, vol. 57, no. 3, pp. 825-882. Chartres: L'Institut, 1950-, 2007.
- [18] Schramm, Oded. Existence and uniqueness of packings with specified combinatorics. *Israel Journal of Mathematics* 73.3 (1991): 321-341.
- [19] Steiner, Jakob. *Systematisch Entwicklung der Abhängigkeit geometrischer Gestalten von Einander, mit Berücksichtigung der Arbeiten alter und neuer Geometer ber Porismen, Projections-Methoden, Geometrie der Lage, etc. Erster Theil*. 1832.
- [20] Steinitz, Ernst. *ber isoperimetrische Probleme bei konvexen Polyedern*. *Journal für die reine und angewandte Mathematik* 159 (1928): 133-143.
- [21] Thurston, William P. *Three-dimensional geometry and topology*. Vol. 1. Princeton university press, 1997.

HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, CHINESE ACADEMIC OF SCIENCES, BEIJING 100190, CHINA

INSTITUTE OF MATHEMATICS, ACADEMIC OF MATHEMATICS & SYSTEM SCIENCES, CHINESE ACADEMIC OF SCIENCES, BEIJING 100190, CHINA

E-mail address: liuj-song@math.ac.cn zhouze@amss.ac.cn